Exercise 7.2.6

In the first-order differential equation $dy/dx = f(x, y)$, the function $f(x, y)$ is a function of the ratio y/x:

$$\frac{dy}{dx} = g(y/x).$$

Show that the substitution of $u = y/x$ leads to a separable equation in u and x.

Solution

Make the change of variables,

$$u = \frac{y}{x}.$$

Then $y = xu$, and the ODE becomes

$$\frac{dy}{dx} = g(u). \tag{1}$$

Use the product rule to evaluate dy/dx:

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Substitute this into equation (1).

$$u + x\frac{du}{dx} = g(u)$$

Bring u to the right side

$$x\frac{du}{dx} = g(u) - u$$

and divide both sides by x.

$$\begin{aligned}\frac{du}{dx} &= \frac{g(u) - u}{x} \\ &= \left(\frac{1}{x}\right)[g(u) - u]\end{aligned}$$

Because du/dx is the product of a function of x and a function of u, this is a separable ODE.